Midterm Introduction to Logic (AI and MA + Guests)

Wednesday 9 December, 2020, 09:00–12:00 AM

1: Translation into propositional logic (10 points) Translate the following sentences into *propositional logic*. Atomic sentences are represented by uppercase letters. Do not forget to provide the translation key — one key for the whole exercise. Represent as much logical structure as possible.

Variant 1a:

a. We will bake a cake only if the oven is already clean or there is oven cleaner at home.

b. It is not the case that both the oven is already clean and there is flour at home.

Variant 1b:

a. We will not bake a cake unless the oven is already clean and there is flour at home.

b. Neither is the oven already clean nor is there oven cleaner at home.

Variant 1c:

a. If there's either no oven cleaner or there is no flour at home, we will not bake a cake.

b. It is not the case that there is no oven cleaner and no flour at home.

Variant 1d:

a. If we will bake a cake, then the oven is already clean and there is flour at home.

b. The oven isn't already clean if and only if there is no oven cleaner at home.

Translation key:

B**:** We will bake a cake.

C**:** The oven is already clean.

O**:** There is oven cleaner at home.

F**:** There is flour at home.

Formalization for version 1a:

a. $B \to (C \lor O)$

b. $\neg(C \land F)$

Formalization for version 1b:

a. $\neg B \lor (C \land F)$
(also correct: $\neg(C \land F) \to \neg B$)

b. $\neg C \land \neg O$ (also correct: $\neg(C \lor O)$)

Formalization for version 1c:

a. $(\neg O \lor \neg F) \to \neg B$

b. $\neg(\neg O \land \neg F)$
(also correct: $O \lor F$)

Formalization for version 1d:

a. $B \to (C \land O)$

b. $\neg C \leftrightarrow \neg O$

2: Translation into first-order logic (10 points) Translate the following sentences to *first-order logic*. Do not forget to provide the translation key — one key for the whole exercise. Represent as much logical structure as possible and let the domain of discourse be the set of all people.

Variant 2a:

a. If Donald does not let Joe talk, then Joe either complains about him, or ignores Donald, but Joe does not lie.

b. Joe lets Donald talk even though Donald does not let Joe talk.

Variant 2b:

a. If Joe complains about Donald, then Donald either does not let Joe talk or he ignores Joe, and he lies.

b. Donald does not let Joe talk even though Joe ignores him.

Variant 2c:

a. If Donald complains about Joe, then he is either not letting Joe talk or he ignores Joe, and he lies.

b. Donald complains about Joe, but Joe ignores him.

Variant 2d:

a. If Donald complains about Joe or he is not letting Joe talk, then Joe ignores him, but if Donald lies, then Joe does not ignore him.

b. Joe ignores Donald and Donald complains about him.

Translation key:

$\mathsf{Talk}(x, y)$**:** x lets y talk.

$\mathsf{Complain}(x, y)$**:** x complains about y.

$\mathsf{Lie}(x)$**:** x lies.

$\mathsf{Ignore}(x, y)$**:** x ignores y.

d**:** Donald

j**:** Joe

Formalizations:

2a)

a. $\neg\mathsf{Talk}(d, j) \rightarrow ((\mathsf{Complain}(j, d) \vee \mathsf{Ignore}(j, d)) \wedge \neg(\mathsf{Complain}(j, d) \wedge \mathsf{Ignore}(j, d)) \wedge \neg\mathsf{Lie}(j))$

b. $\mathsf{Talk}(j, d) \wedge \neg\mathsf{Talk}(d, j)$

2b)

a. $\mathsf{Complain}(j, d) \rightarrow ((\neg\mathsf{Talk}(d, j) \vee \mathsf{Ignore}(d, j)) \wedge \neg(\neg\mathsf{Talk}(d, j) \wedge \mathsf{Ignore}(d, j)) \wedge \mathsf{Lie}(d))$

b. $\neg\mathsf{Talk}(d, j) \wedge \mathsf{Ignore}(j, d)$

2c)

a. $\mathsf{Complain}(d, j) \rightarrow ((\neg\mathsf{Talk}(d, j) \vee \mathsf{Ignore}(d, j)) \wedge \neg(\neg\mathsf{Talk}(d, j) \wedge \mathsf{Ignore}(d, j)) \wedge \mathsf{Lie}(d))$

b. $\mathsf{Complain}(d, j) \wedge \mathsf{Ignore}(j, d)$

2d)

a. $((\mathsf{Complain}(d, j) \vee \neg\mathsf{Talk}(d, j)) \rightarrow \mathsf{Ignore}(j, d)) \wedge (\mathsf{Lies}(d) \rightarrow \neg\mathsf{Ignore}(j, d))$

b. $\mathsf{Ignore}(j, d) \wedge \mathsf{Complain}(d, j)$

3: Formal proofs (30 points) Give formal proofs of the following inferences. Do not forget to provide justifications in the correct order. You may only use the Introduction and Elimination rules and the Reiteration rule.

Variant 3a:

a. $$\begin{array}{|l} a = b \\ b = c \\ d \neq c \\ \hline d \neq a \end{array}$$

b. $$\begin{array}{|l} \\ \hline (A \rightarrow \bot) \leftrightarrow (A \rightarrow \neg A) \end{array}$$

c. $$\begin{array}{|l} A \vee B \\ A \rightarrow \neg C \\ \hline \neg B \rightarrow \neg C \end{array}$$

Variant 3b:

a. $$\begin{array}{|l} P \vee Q \\ P \rightarrow \neg R \\ \hline \neg Q \rightarrow \neg R \end{array}$$

b. $$\begin{array}{|l} a \neq b \\ b = c \\ c = d \\ \hline a \neq d \end{array}$$

c. $$\begin{array}{|l} \\ \hline (P \rightarrow \bot) \leftrightarrow (P \rightarrow \neg P) \end{array}$$

Variant 3c:

a. $$\begin{array}{|l} \\ \hline (\neg A \rightarrow A) \leftrightarrow (\neg A \rightarrow \bot) \end{array}$$

b. $$\begin{array}{|l} P \rightarrow \neg R \\ P \vee Q \\ \hline \neg Q \rightarrow \neg R \end{array}$$

c. $$\begin{array}{|l} a \neq b \\ b = c \\ c = d \\ \hline a \neq d \end{array}$$

a.

1. $a = b$	
2. $b = c$	
3. $d \neq c$	
$\quad$ 4. $d = a$	
$\quad$ 5. $d = b$	=Elim: 4, 1
$\quad$ 6. $d = c$	=Elim: 5, 2
$\quad$ 7. $\perp$	$\perp$ Intro: 6,3
8. $d \neq a$	$\neg$ Intro: 4-7

The proofs involving $=$ and $\neq$ in variants b and c can be done shorter:

1. $a \neq b$	
2. $b = c$	
3. $c = d$	
4. $a \neq c$	=Elim: 1, 2
5. $a \neq d$	=Elim: 4, 3

b.

$\quad$ 1. $A \to \perp$	
$\quad\quad$ 2. A	
$\quad\quad$ 3. $\perp$	$\to$Elim: 1,2
$\quad\quad$ 4. $\neg A$	$\perp$Elim: 3
$\quad$ 5. $A \to \neg A$	$\to$Intro: 2–4
$\quad$ 6. $A \to \neg A$	
$\quad\quad$ 7. A	
$\quad\quad$ 8. $\neg A$	$\to$Elim: 6,7
$\quad\quad$ 9. $\perp$	$\perp$Intro: 7,8
$\quad$ 10. $A \to \perp$	$\to$Intro: 7–9
11. $(A \to \perp) \leftrightarrow (A \to \neg A)$	$\leftrightarrow$ Intro: 1–5, 6–10

c.

1. $A \vee B$	
2. $A \to \neg C$	
$\quad$ 3. $\neg B$	
$\quad\quad$ 4. A	
$\quad\quad$ 5. A	Reit: 4
$\quad\quad$ 6. B	
$\quad\quad$ 7. $\perp$	$\perp$ Intro: 6, 3
$\quad\quad$ 8. A	$\perp$ Elim: 7
$\quad$ 9. A	$\vee$ Elim: 1, 4–5, 6–8
$\quad$ 10. $\neg C$	$\to$ Elim: 2, 9
11. $\neg B \to \neg C$	$\to$ Intro: 3–10

Variants 3b and 3c are notational variants with premises reshuffled. In variant 3c, the terms at both ends of the bi-implication have been swapped, and $\neg A$ and A have been interchanged.

4: Truth tables (15 points) Use *truth tables* to answer the next questions.

Variant 4a:

a. Is $(\neg(P \wedge \neg Q) \rightarrow (Q \rightarrow \neg R)) \leftrightarrow (P \vee \neg R)$ a *tautology*?
Do not forget to draw an explicit conclusion from the truth table and to explain your answer.

b. Is the sentence $\neg(a = c) \vee (\neg(b = a) \rightarrow \neg(b = c))$ a *logical truth*?
Draw an explicit conclusion, explain your answer and indicate the spurious rows.

Variant 4b:

a. Is the sentence $\neg(a = c) \vee (\neg(b = a) \rightarrow \neg(b = c))$ a *logical truth*?
Draw an explicit conclusion, explain your answer and indicate the spurious rows.

b. Is $(\neg(A \wedge \neg B) \rightarrow (B \rightarrow \neg C)) \leftrightarrow (A \vee \neg C)$ a *tautology*?
Do not forget to draw an explicit conclusion from the truth table and to explain your answer.

tautology?

P	Q	R	$(\neg(P$	$\wedge$	$\neg Q)$	$\rightarrow$	$(Q$	$\rightarrow$	$\neg R))$	$\leftrightarrow$	$(P$	$\vee$	$\neg R)$
T	T	T	T	F	F	F		F	F	**F**		T	F
T	T	F	T	F	F	T		T	T	**T**		T	T
T	F	T	F	T	T	T		T	F	**T**		T	F
T	F	F	F	T	T	T		T	T	**T**		T	T
F	T	T	T	F	F	F		F	F	**T**		F	F
F	T	F	T	F	F	T		T	T	**T**		T	T
F	F	T	T	F	T	T		T	F	**F**		F	F
F	F	F	T	F	T	T		T	T	**T**		T	T

The sentence *is not* a tautology, because there are some 'F's under the main connective in rows 1 and 7.

logical truth? Variant a:

$a = c$	$b = a$	$b = c$	$\neg$	$(a = c)$	$\vee$	$(\ \neg$	$(b = a)$	$\rightarrow$	$\neg$	$(b = c))$	
T	T	T	F	T	**T**	F	T	T	F	T	
T	T	F	F	T	**T**	F	T	T	T	F	spurious!
T	F	T	F	T	**F**	T	F	F	F	T	spurious
T	F	F	F	T	**T**	T	F	T	T	F	
F	T	T	T	F	**T**	F	T	T	F	T	spurious!
F	T	F	T	F	**T**	F	T	T	T	F	
F	F	T	T	F	**T**	T	F	F	F	T	
F	F	F	T	F	**T**	T	F	T	T	F	

The sentence *is* a logical truth because the only 'F' under the main connective in a spurious row, namely in row 3.

Variant b:

$b=c$	$b=a$	$a=c$	$\neg$	$(a=c)$	$\lor$	$(\neg$	$(b=a)$	$\to$	$\neg$	$(b=c))$	
T	T	T	F	T	**T**	F	T	T	F	T	
T	T	F	T	F	**T**	F	T	T	F	T	spurious!
T	F	T	F	T	**F**	T	F	F	F	T	spurious
T	F	F	T	F	**T**	T	F	F	F	T	
F	T	T	F	T	**T**	F	T	T	T	F	spurious!
F	T	F	T	F	**T**	F	T	T	T	F	
F	F	T	F	T	**T**	T	F	T	T	F	
F	F	F	T	F	**T**	T	F	T	T	F	

The sentence *is* a logical truth because the only 'F' under the main connective in a spurious row, namely in row 3.

Variant 4b is a notational variant, with the order of the questions swapped. Note that in variant 4b (a), the order of the columns has been swapped, so the truth table is slightly different.

5: Normal forms of propositional logic (15 points)

Variant 5a:

a. Provide a negation normal form (NNF) of this sentence: $((P \lor \neg Q) \lor R) \leftrightarrow (\neg R \land Q)$.

b. Provide a conjunctive normal form (CNF) of this sentence:

$$P \to \neg(\neg Q \lor \neg(R \land \neg S))$$

Variant 5b:

a. Provide a conjunctive normal form (CNF) of this sentence:

$$A \to \neg(\neg B \lor \neg(C \land \neg D))$$

b. Provide a negation normal form (NNF) of this sentence: $((A \lor \neg B) \lor C) \leftrightarrow (\neg C \land B)$.

a.

$$\begin{array}{lll}
 & (((P \lor \neg Q) \lor R) \leftrightarrow (\neg R \land Q)) & \\
\Leftrightarrow & (((P \lor \neg Q) \lor R) \to (\neg R \land Q)) \land ((\neg R \land Q) \to ((P \lor \neg Q) \lor R)) & \text{def } \leftrightarrow \\
\Leftrightarrow & (\neg((P \lor \neg Q) \lor R) \lor (\neg R \land Q)) \land ((\neg R \land Q) \to ((P \lor \neg Q) \lor R)) & \text{def } \to \\
\Leftrightarrow & ((\neg(P \lor \neg Q) \land \neg R) \lor (\neg R \land Q)) \land ((\neg R \land Q) \to ((P \lor \neg Q) \lor R)) & \text{De Morgan} \\
\Leftrightarrow & (((\neg P \land \neg\neg Q) \land \neg R) \lor (\neg R \land Q)) \land ((\neg R \land Q) \to ((P \lor \neg Q) \lor R)) & \text{De Morgan} \\
\Leftrightarrow & (((\neg P \land Q) \land \neg R) \lor (\neg R \land Q)) \land ((\neg R \land Q) \to ((P \lor \neg Q) \lor R)) & \text{double } \neg \\
\Leftrightarrow & (((\neg P \land Q) \land \neg R) \lor (\neg R \land Q)) \land (\neg(\neg R \land Q) \lor ((P \lor \neg Q) \lor R)) & \text{def } \to \\
\Leftrightarrow & (((\neg P \land Q) \land \neg R) \lor (\neg R \land Q)) \land ((\neg\neg R \lor \neg Q) \lor ((P \lor \neg Q) \lor R)) & \text{De Morgan} \\
\Leftrightarrow & (((\neg P \land Q) \land \neg R) \lor (\neg R \land Q)) \land ((R \lor \neg Q) \lor ((P \lor \neg Q) \lor R)) & \text{double } \neg
\end{array}$$

b.

$$\begin{array}{rll}
& P \to \neg(\neg Q \vee \neg(R \wedge \neg S)) & \\
\Longleftrightarrow & \neg P \vee \neg(\neg Q \vee \neg(R \wedge \neg S)) & \text{def} \to \\
\Longleftrightarrow & \neg P \vee (\neg\neg Q \wedge \neg\neg(R \wedge \neg S)) & \text{De Morgan} \\
\Longleftrightarrow & (\neg P \vee \neg\neg Q) \wedge (\neg P \vee \neg\neg(R \wedge \neg S)) & \text{distributivity} \\
\Longleftrightarrow & (\neg P \vee Q) \wedge (\neg P \vee \neg\neg(R \wedge \neg S)) & \text{double } \neg \\
\Longleftrightarrow & (\neg P \vee Q) \wedge (\neg P \vee (R \wedge \neg S)) & \text{double } \neg \\
\Longleftrightarrow & (\neg P \vee Q) \wedge ((\neg P \vee R) \wedge (\neg P \vee \neg S)) & \text{distributivity}
\end{array}$$

Variant 5b is a notational variant, with the order of the questions swapped and different letters used.

6: Set theory (10 points)

Variant 6a:

Consider the following sets:

$$A = \{a, \{b, c\}\}, \;\; B = \{\emptyset, \{b, c\}\}, \;\; C = \{b, c\}, \;\; D = \{a\} \;\; \text{and} \;\; R = \{\langle a, b\rangle, \langle a, c\rangle\}.$$

Provide a list of five true and a list of five false statements in the language of set theory, clearly indicating which statements are true and which are false.

You should use each of the sets above at least twice and each of the following symbols at least once.

$$\cap \qquad \cup \qquad \setminus \qquad \subseteq \qquad \subsetneq \qquad \times \qquad \emptyset \qquad \in$$

Variant 6b:

Consider the following sets:

$$A = \{b, \{c, a\}\}, \;\; B = \{\emptyset, \{c, a\}\}, \;\; C = \{c, a\}, \;\; D = \{b\} \;\; \text{and} \;\; R = \{\langle b, c\rangle, \langle b, a\rangle\}.$$

Provide a list of five true and a list of five false statements in the language of set theory, clearly indicating which statements are true and which are false.

You should use each of the sets above at least twice and each of the following symbols at least once.

$$\cap \qquad \cup \qquad \setminus \qquad \subseteq \qquad \subsetneq \qquad \times \qquad \emptyset \qquad \in$$

7: Bonus question (10 points) Give a formal proof of the following inference:

Variant 7a:

$$\begin{array}{|l}
P \vee R \\
\hline
(\neg P \leftrightarrow R) \vee (P \wedge R)
\end{array}$$

Variant 7b:

$$\begin{array}{|l}
A \vee B \\
\hline
(A \wedge B) \vee (\neg A \leftrightarrow B)
\end{array}$$

1	$P \vee R$	
2.	$\neg((\neg P \leftrightarrow R) \vee (P \wedge R))$	
3	P	
4	R	
5	$P \wedge R$	$\wedge$ Intro : 3, 4
6	$(\neg P \leftrightarrow R) \vee (P \wedge R)$	$\vee$ Intro : 5
7	$\perp$	$\perp$ Intro : 6, 2
8	$\neg R$	$\neg$ Intro : 4-7
9	$\neg P$	
10	$\perp$	$\perp$ Intro : 3, 9
11	R	$\perp$ Elim : 10
12	R	
13	$\perp$	$\perp$ Intro : 12, 8
14	$\neg P$	$\perp$ Elim : 13
15	$\neg P \leftrightarrow R$	$\leftrightarrow$ Intro : 9-11, 12-14
16	$(\neg P \leftrightarrow R) \vee (P \wedge R)$	$\vee$ Intro : 15
17	$\perp$	$\perp$ Intro : 16, 2
18	R	
19	P	
20	$P \wedge R$	$\wedge$ Intro : 19, 18
21	$(\neg P \leftrightarrow R) \vee (P \wedge R)$	$\vee$ Intro : 20
22	$\perp$	$\perp$ Intro : 21, 2
23	$\neg P$	
24	$\neg P$	
25	R	Reit : 18
26	R	
27	$\neg P$	Reit : 23
28	$(\neg P \leftrightarrow R)$	$\leftrightarrow$ Intro : 24-25, 26-27
29	$(\neg P \leftrightarrow R) \vee (P \wedge R)$	$\vee$ Intro : 28
30	$\perp$	$\perp$ Intro : 29, 2
31	$\perp$	$\vee$ Elim : 3-17, 18-30
32	$\neg\neg((\neg P \leftrightarrow R) \vee (P \wedge R))$	$\neg$ Intro : 2-31
33.	$(\neg P \leftrightarrow R) \vee (P \wedge R)$	$\neg$ Elim : 32